(2) $f(z) = |z|^2 = x^2 + y^2 + i\cdot 0$. Note $u(x,y) = x^2+y^2$ and $v(x,y) = 0$. These have continuous first order partial derivatives in a neighborhood of 0:

$$u_x = 2x \qquad v_x = 0$$
$$u_y = 2y \qquad v_y = 0.$$

The Cauchy Riem. eq' are satisfied at 0, so $f'(0) \overset{=0}{}$ exists. //

The necessary and sufficient conditions can be translated into polar coordinates. If $f(z) = u(r,\theta) + i\, v(r,\theta)$, then the polar form of the CR-equations at $z = re^{i\theta}$ is

$$\boxed{\begin{cases} r u_r = v_\theta \\ u_\theta = -r v_r \end{cases} \quad \left(\text{evaluated at } (r,\theta)\right)}$$

and the value of $f'(z)$ is

$$f'(re^{i\theta}) = e^{-i\theta}\left(u_r(r,\theta) + i\, v_r(r,\theta)\right).$$

The precise statements of these theorems are in the book. //

$\boxed{\text{Example}}$ Consider

$$f(z) = \sqrt{r}\, e^{i\frac{\theta}{2}}, \quad (r>0, -\pi < \theta < \pi)$$

This is the function that takes values as the principal square root of $z$. We compute $f'$ using the sufficient condition in polar form. Notice

$$f(z) = \underbrace{\sqrt{r}\cos\theta/2}_{u(r,\theta)} + i\,\underbrace{\sqrt{r}\sin\theta/2}_{v(r,\theta)}.$$

Note

$$r\, u_r(r,\theta) = r\frac{1}{2\sqrt{r}}\cos\theta/2 = v_\theta(r,\theta)$$

$$u_\theta(r,\theta) = -\frac{1}{2}\sqrt{r}\sin\theta/2 = -r\, v_r(r,\theta)$$

So Cauchy-Riemann are satisfied everywhere and the partial are continuous everywhere. Then

$$f' = e^{-i\theta}[u_r + iv_r$$
$$= e^{-i\theta}\left(\frac{1}{2\sqrt{r}}\cos\theta/2 + i\frac{1}{2\sqrt{r}}\sin\theta/2\right)$$
$$= \frac{1}{2\sqrt{r}}e^{-i\theta}\left(\cos\theta/2 + i\sin\theta/2\right) = \frac{1}{2\sqrt{r}}e^{-i\theta/2} = \frac{1}{2f(z)}. \quad //$$

# Analytic Functions

**Definition (Analytic)** A function $f$ is analytic on an open set $U$ if $f'(z)$ exists for all $z \in U$. We say that $f$ is analytic at a point $z_0$ if it is analytic some open disk $D_\varepsilon(z_0)$. A

function is entire if it is analytic on $\mathbb{C}$. //

**Example**

(1) $f(z) = 1/z$ is analytic on any open set not containing $0$, in particular on $\mathbb{C}\setminus\{0\}$.

(2) $f(z) = |z|^2$ is not analytic anywhere since we showed that $f'(z)$ exists if and only if $z=0$.

(3) Polynomials are entire. //

Let $D$ be a domain. We know several necessary or sufficient for a function $f$ to be analytic on $D$:

(Necessary) (1) $f$ is continuous on all of $D$.

(2) Cauchy-Riemann eq. satisfied on $D$.

(sufficient) (1) Cauchy-Riemann eq satisfied on $D$ and the partial derivatives of $u$ and $v$ are continuous on all of $D$.

(2) The differentiation rules. If $f$ and $g$ are analytic on $D$, then so are $f+g$, $fg$, $f/g$ ($g \neq 0$ on $D$).

(3) The composition of analytic functions is analytic.

$\boxed{\text{Theorem (sufficient condition for } f \text{ to be constant)}}$ Suppose that $D$ is a domain and $f'(z) = 0$ for all $z \in D$. Then $f(z)$ is constant on $D$.

Proof. Write $f(z) = u(x,y) + i v(x,y)$. We have

$$0 = f'(z) = u_x + i v_x$$

$$0 = f'(z) = v_y - i u_y$$

Hence, $u_x = u_y = 0$ and $v_x = v_y = 0$. Let $L$ be any line segment connected points $p, q \in D$. Let $\vec{w} = (a,b)$ be a unit vector parallel to $L$. The directional derivative of $u$ along $L$ is

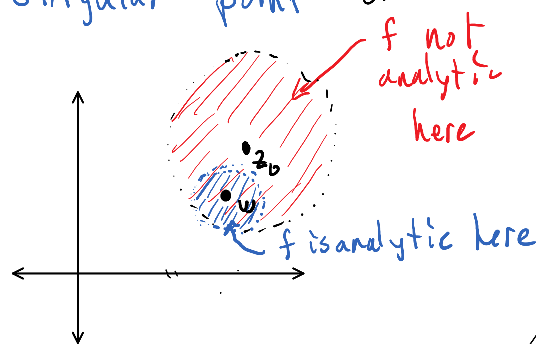
$$\nabla u \cdot \vec{w} = a u_x + b u_y = 0.$$

So $u$ is constant on $L$. Since $D$ is a domain, any two points can be connected by a polygonal line. Apply this argument to each line segment to see that $u$ has the

the same value at the endpoints. This shows that $u$ is constant on $D$. The same argument works for $v$. Hence, if $u(x,y) = c$ and $v(x,y) = d$, then

$$f(z) = c + id.$$

$\boxed{\text{Definition (singularities)}}$ If $f$ is not analytic at $z_0$, but every neighborhood of $z_0$ contains a point at which $f$ is analytic, then $z_0$ is called a singular point or singularity of $f$.




$\boxed{\text{Example}}$

(1) $f(z) = 1/z$ has a singularity at $0$.

(2) $f(z) = |z|^2$ has no singular points. $f$ is not analytic at $0$, but any open neighborhood of $0$ contains no points at which $f$ is analytic.

(3) $f(z) = \dfrac{z^2+3}{(z+1)(z^2+5)}$ has singularities when

$$(z+1)(z^2+5) = 0$$

i.e. if $z = -1, i\sqrt{5}, -i\sqrt{5}$.

(4) $f(z) = \sin x \cosh y + i \cos x \sinh y$ is entire and hence has no singularities. In fact, $f$ has derivatives

of all orders.

Note that $u(x,y) = \sin x \cosh y$ and $v(x,y) = \cos x \sinh y$.

Recall:

$$\begin{cases} \sinh y = \dfrac{e^{y} - e^{-y}}{2} & \dfrac{d}{dy}\sinh y = \cosh y \\ \cosh y = \dfrac{e^{y}+e^{-y}}{2} & \dfrac{d}{dy}\cosh y = \sinh y. \end{cases}$$

Then
$$u_x(x,y) = \cos x \cosh y = v_y$$
$$u_y(x,y) = \sin x \sinh y = -v_x$$

So the CR-eq are satisfied everywhere and the partials are continuous everywhere. So $f'$ exists everywhere and

$$f'(z) = u_x + i v_x = \underbrace{\cos x \cosh y}_{U(x,y)} - i \underbrace{\sin x \sinh y}_{V(x,y)}.$$

This shows that $f$ is entire and has no singularities.

Then
$$U_x(x,y) = -\sin x \cosh y = V_y$$
$$U_y(x,y) = \cos x \sinh y = -V_x.$$
So $f''(z)$ exists

and
$$f''(z) = U_x + i V_x = -\sin x \cosh y - i \cos x \sinh y$$
$$= -f(z).$$

So $f'$ is entire and $f'' = -f(z)$. But $f$ is entire! You can see now that we can differentiate to all orders. //

**Proposition** Suppose $f$ and $\bar{f}$ are analytic on a domain $D$. Then $f$ is constant on $D$.

Proof. Write
$$f(z) = u(x,y) + i v(x,y)$$
$$\bar{f}(z) = u(x,y) - i v(x,y).$$

Then $f, \bar{f}$ satisfy CR-eq on $D$:

for $f$ : $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ for $\bar{f}$ : $\begin{cases} u_x = -v_y \\ u_y = v_x \end{cases}$

Hence, $u_x = v_y = -u_x$ so $u_x = 0$. Also, $v_x = -u_y = -v_x$, so $v_x = 0$. Now, $f'(z) = u_x + i v_x = 0$. By the theorem, $f$ is constant on $D$. ∎

**Proposition** Suppose that $f$ is analytic on a domain $D$ and $|f(z)|$ is constant on $D$. Then $f$ is constant on $D$.

Proof. Assume

$$(*) \quad |f(z)| = c \qquad \text{for all } z \in D.$$

then

$$(*) \quad f(z)\bar{f}(z) = |f(z)|^2 = c^2 .$$

Suppose $c = 0$. Then by $(*)$ $f = 0$ so $f$ is constant. If $c \neq 0$, then by $(*)$ $f(z) \neq 0$ for all $z \in D$. So then

$$\bar{f}(z) = \frac{c^2}{f(z)} .$$

But $\frac{c^2}{f(z)}$ is analytic since $f$ is. Hence, $\bar{f}$ is analytic. Hence, $f$ is constant by the preceding proposition. ∎

# Chapter 3: Elementary Functions

The goal of this chapter is to define analytic functions of a complex variable $z$ that reduce to the elementary functions studied in calculus when $z$ is real. Namely,

(1) exponential functions

(2) logarithms

(3) power functions

(4) trig functions + their inverses

(5) hyperbolic trig functions.

We will also develop their basic properties.

# Exponential Function

**Definition (The Exponential Function)** The exponential function $e^z$ or $\exp z$ is defined on $\mathbb{C}$ by the formula

$iy \in \mathbb{C}\setminus\mathbb{R}$ so this eulers formula.

$$e^z \stackrel{\text{def}}{=} e^x e^{iy} = e^x \cos y + i e^x \sin y \qquad (z = x+iy).$$

$x \in \mathbb{R}$ so this is the usual exponential

Note: if $z = x \in \mathbb{R}$, then $e^z = e^x$ is the usual exponential.

**Proposition (Properties of the exponential)** Let $z, w \in \mathbb{C}$.

(1) $|e^z| = e^x$ and $\arg e^z = y + 2k\pi$, $k \in \mathbb{Z}$

(2) $e^{z+w} = e^z e^w$

(3) $e^{z-w} = e^z / e^w$

(4) $e^z$ is entire and $\frac{d}{dz} e^z = e^z$

(5) $e^z$ is periodic: $e^{z+2k\pi i} = e^z$ for all $k \in \mathbb{Z}$.

Proof.

(1) By definition $e^z = e^x e^{iy}$ is in exponential form so $|e^z| = e^x$ and $\arg e^z = y + 2k\pi$, $k \in \mathbb{Z}$.

(2) Write $z = x+iy$ and $w = u+iv$. Then

$$e^{z+w} = e^{(x+u)+i(y+v)} = e^{x+u} e^{i(y+v)} = e^x e^u e^{iy} e^{iv}$$
$$= e^x e^{iy} e^u e^{iv}$$
$$= e^z e^w.$$

(3) Follows from (2) since

$$e^{z-w} e^w \stackrel{(2)}{=} e^z.$$

(4) We proved that $e^z$ is entire in an example. $\frac{d}{dz} e^z = e^x$ follows from $f' = u_x + i v_x$.

(5) Follows from (2):

$$e^{z+2k\pi i} \overset{(2)}{=} e^z e^{2k\pi i} = e^z. \qquad \square$$

## Logarithms

The logarithmic function arises when solving the equation

$$e^w = z \qquad (z \neq 0)$$

for $w$. Write $z = re^{i\theta}$ and $w = u + iv$. Then

$$e^u e^{iv} = e^w = z = re^{i\theta}.$$

So $e^u = r$ and $v = \underbrace{\theta + 2k\pi}_{\arg z}$, $k \in \mathbb{Z}$.

$\hookrightarrow u = \ln r = \ln|z|$. So $w = \ln|z| + i \arg z$.

natural log

**Definition (logarithmic function)** Following the computation, we define the logarithmic function $\log z$ for any $z \neq 0$ via

$$\log z \overset{\text{def}}{=} \ln|z| + i \arg z.$$

Note: $\log z$ is multiple-valued. The principal branch of $\log z$ is denoted $\operatorname{Log} z$ and is defined by taking the principal argument of $z$:

$$\operatorname{Log} z \overset{\text{def}}{=} \ln|z| + i \operatorname{Arg} z.$$

The principal branch of log is a single-valued function. //

**Proposition (Properties of log / Log)**

(1) $e^{\log z} = z$

(2) $\log e^z = z + 2k\pi i$, $k \in \mathbb{Z}$.

(3) $\log z = \operatorname{Log} z + 2k\pi i, \quad k \in \mathbb{Z}$

(4) If $z = x$ is a positive real number, then $\operatorname{Log} z = \ln x$.

Proof.

(1) $e^{\log z} = e^{\ln|z| + i \arg z} = e^{\ln|z| + i(\operatorname{Arg} z + 2k\pi)}$

$= e^{\ln|z|} \cdot e^{i \operatorname{Arg} z} \cdot e^{2k\pi i}$

$= |z| \, e^{i \operatorname{Arg} z} = z.$

(2) $\log e^z = \ln|e^z| + i \arg(e^z)$

$= \ln e^x + i(y + 2k\pi) \quad k \in \mathbb{Z}, \quad z = x + iy$

$\cancel{=} \; x + iy + 2k\pi i$

$= z + 2k\pi i$

(3) $\operatorname{Log} z + 2k\pi i = \ln|z| + i \operatorname{Arg} z + 2k\pi i \qquad (k \in \mathbb{Z})$

$= \ln|z| + i(\operatorname{Arg} z + 2k\pi) \qquad (k \in \mathbb{Z})$

$= \ln|z| + i(\arg z)$

$= \log z.$

(4) If $z = x > 0$, then $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$

$= \ln x.$ ∎